Grade 8 Mathematics Lesson Plan

June 23

Date: Tuesday, June 23, 2015
Period 6 (14:20 - 15:10)
Classroom: Grade 8 Homeroom D
40 students (20 boys and 20 girls)
Teacher: SHIBATA, Sho

0 Research theme and rationale for the lesson

0.1 Research theme

0.1.1 Statement of research theme

Lessons that raise the quality of mathematical process.

0.1.2 About research theme

This research lesson is a part of research by the Mathematics Education Research Group of Tokyo Gakugei attached secondary schools. The purpose of this research group is as follows. Although mathematics education in Japan has emphasized the importance of the processes of mathematical exploration and the development of ways of thinking and abilities that are needed in those processes, mathematics teaching still focus almost exclusively on teaching of mathematical content. Of course, we are not saying that mathematical content is unimportant. However, even though "mathematical ways of observing and thinking" and "mathematical activities" have been long stressed in Japan, we wonder if mathematics lessons in Japanese classrooms clearly reflect the emphases.

To emphasize mathematical processes means to emphasize the process of using and creating mathematics. As an activity, that process may be grasped as "mathematical activity." The ways of observing and thinking utilized in that process is "mathematical ways of observing and thinking. In this way, our research group considers the processes involved in using and creating mathematics as "mathematical process," and we are exploring how mathematics lessons ought to raise the quality of "mathematical process." Therefore, our research theme is "lessons that raise the quality of mathematical process."

I believe that "raising the quality of mathematical process" is the same as raising the quality of students mathematics learning. In other words, to use and create mathematics means to approach solutions for mathematical problems. To raise its quality therefore means to raise the quality of problem solving processes such as asking "is there a better approach?" "can this method be generalized?" or "is it possible to use strategies we have learned previously?" Of course, the quality of mathematical process does not necessarily rise in a single lesson. Rather, the quality will rise as students study a single topic, a unit, and the entire school mathematics curriculum.

How should teachers organize a unit and develop each individual lessons in the unit in order to "raise the quality of mathematical process"? What kinds of tasks should they use, and how should they interact with students? I believe the key to answer these questions is in the lessons based on problem solving. In such lessons, ideas from students' independent problem solving activities are shared with the whole class, and through the whole class discussion and appropriate guidance from the teacher, simple or incomplete mathematical ideas are molded into more refined mathematical ideas and processes. The ultimate goal is for students who studied mathematics to be able to interiorize others in mathematics lessons and think about a variety of ways to solve a problem, identify what is truly important by examining those ideas and further expand their ideas. I believe that this ultimate goal can only be achieved by continually and intentionally setting up problem solving based lessons.

1 Name of the Unit: Calculations with Algebraic Expressions[1]

2 Goals of the Unit

- Students will enjoy mathematical activities and merits of thinking mathematically through activities such as identifying patterns and relationships by making use of variables and algebraic expressions. They will try to utilize algebraic expressions in problem solving and mathematical exploration. [Interest, Eagerness, and Attitude]
- Students will be able to represent patterns and relationships of quantities in phenomena using algebraic expressions. They will be able to explain by making use of generalization. Furthermore, students will understand that algebraic expressions represent both the process and the result of calculations by interpreting or evaluating the given algebraic expressions. They will be able to interpret specific phenomenon or relationships expressed in algebraic expressions. [Mathematical Way of Thinking]
- Students will be able to add and subtract simple polynomials as well as multiply and divide monomials. They can represent and interpret patterns and relationships of quantities in algebraic expressions. [Mathematical Skills]
- Students will understand the meaning and purpose of algebraic expressions. [Knowledge and Understanding]

3 About mathematics teaching

In elementary school, students have used symbols □ and △ to represent the relationships between addition and subtraction or multiplication and division in equations such as 5＋□＝8, 3×△＝24. They have also used equations with words such as (Speed) × (Time) = (Distance) to represent relationships among quantities. In lower secondary school so far, as the foundation for the study of algebraic expressions, they have learned that letters such as a and x can be used in place of words or symbols such as □ and △. They have also learned to use algebraic expressions to grasp direct and inverse proportional relationships. Moreover, they have also used quasi-variables as they thought about division of fractions or representing numerical relationships.

In Grade 7, in addition to representing quantities using letters and manipulating expressions, students have learned about different ways letters may be used, as unknowns, as variables, or as the representitive of a set. Moreover, instead of procedurally introducing letters and practicing manipulation of algebraic expressions, we first examined numbers that act like variables (quasi-variables). Using quasi-

[1] Translator Comment: The Japanese word used here in the original is *shiki* which includes both expressions and equations.

variables, we interpreted the meaning of an expression and its structure. We introduced letters in the process of generalization.

In Grade 8, students can examine even wider variety of situations using more letters in algebraic expressions. At the same time, when there are 2 or more variables in a situation, students must think carefully about the relationship between the variables. For example, consider the statement, "the sum of two consecutive even numbers is always an even number." In order to represent the two even numbers, students must think whether they should use a whole number n and represent them as $(2n, 2n + 2)$, or use whole numbers n and m and represent them as $(2m, 2n)$. The representation $(2n, 2n + 2)$ is the ideal approach, but since the representation $(2m, 2n)$ has fewer constraints, we cannot say it is totally incorrect as a proof of the statement. On the other hand, the relationship that the sum of 2 consecutive even numbers is the double of the odd number in between them can be interpreted from the representation $(2n, 2n + 2)$ only. These ideas are not only useful in understanding conventions about algebraic expressions but also raising students' understanding of algebraic expressions and variables to a higher level. As a result, students can represent more and more phenomena mathematically, transform algebraic expressions and generate new insights. It is also possible to further expand students' study of mathematics based on their own solutions by critically reflecting on them and polishing their written solutions.

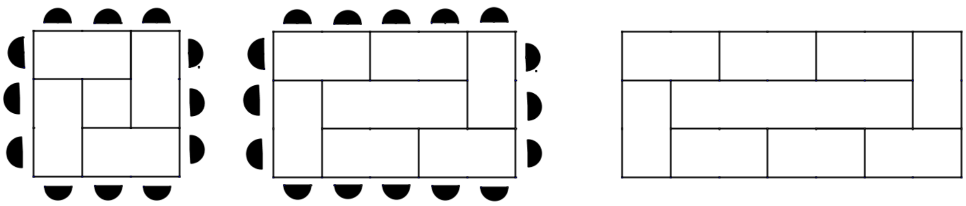
4. About students

Last year, these students have studied simple algebraic expressions, linear equations, and direct and inverse proportion relationships. As they studied those topics, they used letters to investigate changes, generalize patterns, represent relationships as functions, and determine unknown quantities. In mathematics lessons, I encouraged students to examine their own problem solving processes by attending to "what ought to be considered," and "perspective upon which my observation is based." So far this year, we have focused on the combination of identifying patterns and justifying the observations using letters through investigations such as divisibility rules and other properties of whole numbers.

Although students are not afraid of mathematics, some students find it difficult to "explain" their ideas. However, from their experiences, they know that their mathematical understanding is deepened when we resolve questions raised by students who are struggling. As a result, I believe there is a classroom culture in which students can openly admit that they do not understand something. Many of the students eagerly share their ideas even when their ideas may appear to be rather simple.

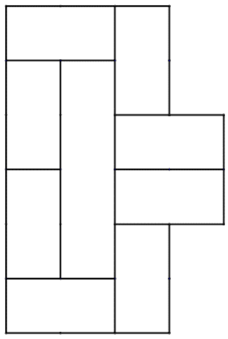
5. About the task

The task in this lesson asks for students to determine the number of tables needed for a given number of people when the tables are arranged as shown below.



The value of this task is that it involves 3 related variables. The 3 variables are the number of tables, the number of people, and the position of the sequence of arrangements, which is implicit in the task. If students realize that there will be 2 people seated at each table, and regardless of the number of tables, there will be 4 people seated at the short side of the table, then they can express the relationship between the number of tables and the number of people. However, I anticipate that there will be students who think, "in the first arrangement, there are 4 tables and 12 people, in the second arrangement ...," using the variable that is not explicitly given in the task. Those students are actually expressing both the number of tables and the number of people as functions of the position numbers. This reasoning is simply replacing two co-varying quantities using a variable that might be easier to manipulate.

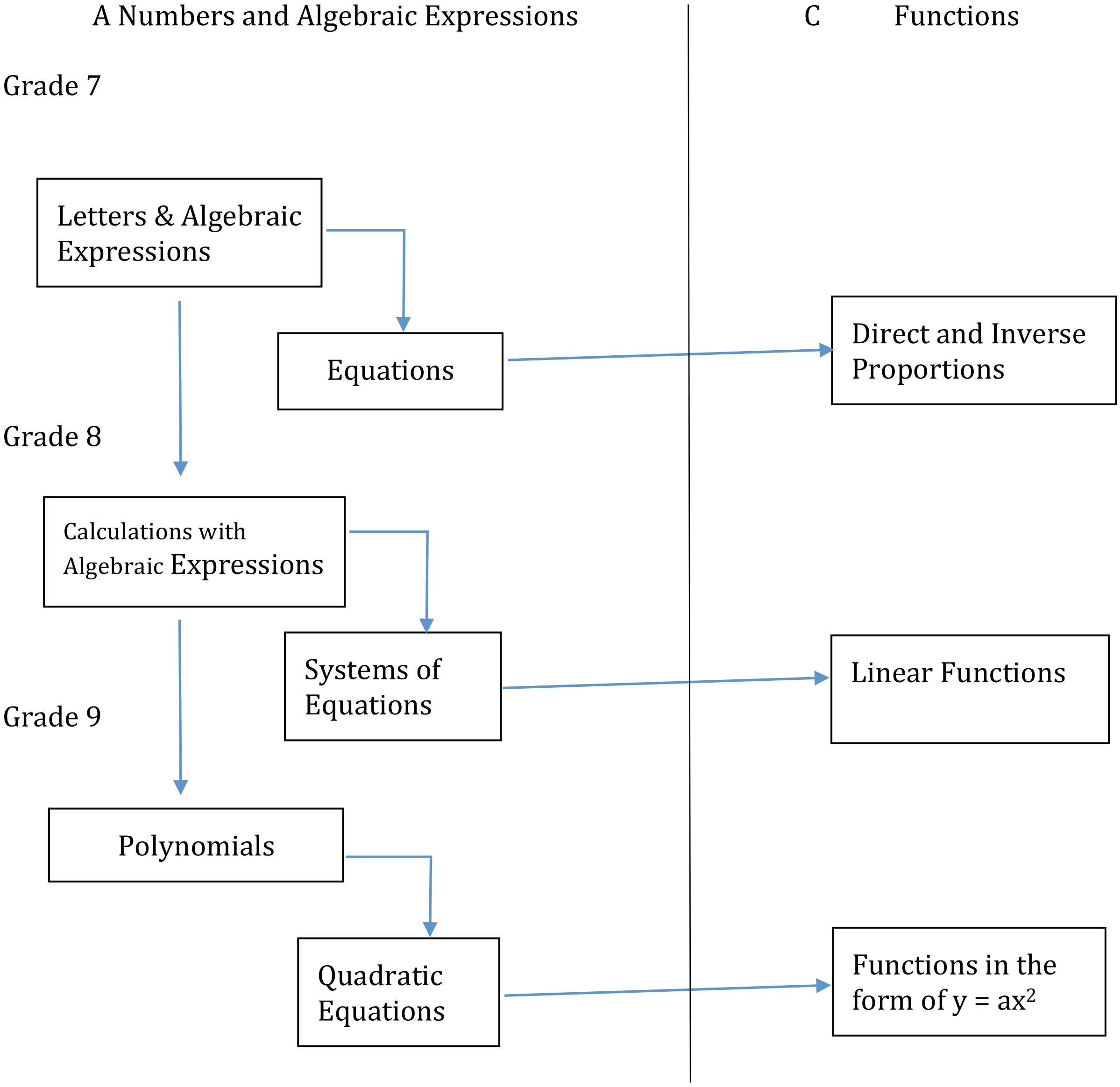
Another interesting point of this task is that the number of people is fixed regardless of how the tables are arranged. For example, if we use 8 tables, the number of people who can be seated will be 20 whether the desks are arranged in an oblong arrangement like the one shown above or in a square. even if the 8 tables are arranged in an unconventional way like what is shown on the right (the short side must be a half of the long side), the number of people will remain 20.



In the whole class discussion after the independent problem solving time, we will critically compare and analyze the approach that tries to determine the relationship between the number of tables and the number of people directly and the approach that uses the position number as a variable. We will then focus on the algebraic equations for these approaches. The equation that show the relationship between the number of tables (t) and the number of people (p) directly will be $p = 2t + 4$. The approach that considers the position number (s) will have two equations, $t = 2s + 2$ and $p = 4s + 8$. We will then examine how these equations are related to each other. As students substitute numbers or even expressions in each variable, students will eventually realize that the same situations is represented in two different ways using different variables.

In the lower secondary school mathematics curricula, students will not learn to substitute expressions into variables until they learn to solve systems of equations using the substitution method. However, while exploring patterns, students sometime realize that two variables are determined by another independent variable, and that realization may deepen their understanding of the situation or even lead to the solution. This way of reasoning is important in learning of algebraic expressions, and it should be discussed as a topic with its own merit, not just as a method of solving systems of equations. How this task will be solved will depend on what variables and their relationships students pay attention to. I believe that the experience of comparing these solution strategies and substituting expressions in letters will make students' learning of the substitution method for solving systems of equations easier. Moreover, students should be able to approach future mathematical explorations more creatively and their mathematical activities can be expanded.

6. Scope and Sequence



7 Unit Plan

Sub-unit 1: Properties of Integers (5 lessons)
Sub-unit 2: Calculations of algebraic expressions (5 lessons)
Sub-unit 3: Explaining with algebraic expressions (10 lessons)
Sub-unit 4: Applications of algebraic expressions (5 lessons)
Today's lessons is the 3rd lesson of this sub-unit.

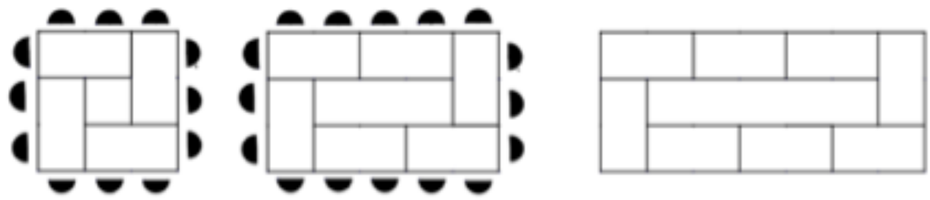
8 Today's lesson

(1) Goals

- Students will be able to identify quantities that are related to the unknown quantity and solve problems by making use of the relationship. [Mathematical Way of Thinking]
- Students will think about relationships and patterns among quantities and try to make sense of the relationship among the number of tables, the way the tables are arranged and the number of people who can be seated. [Interest, Eagerness, and Attitude]

(2) Flow of the lesson

<table>
<tr><td>Main Learning Activity</td><td>Anticipated Students' Responses</td><td># Instructional considerations
○ Evaluation</td></tr>
<tr><td>[Introduction]</td><td></td><td></td></tr>
<tr><td colspan="3">Problem
There are tables as shown on the right. Two people can be seated on the long side and one person can be seated on the short side.

If we arrange the tables as shown below, how many tables do we need to seat 75 people?
</td></tr>
<tr><td></td><td>• Can people be seated inside?
• Is it only getting longer to the side?</td><td># Display a pre-printed diagrams so that they will not restrict students' thinking.</td></tr>
</table>

[Development]

- Independent problem solving

(Using s for the position number, t for the number of tables, and p for the number of people)

(1) By considering s an independent variable and t and p dependent variables, determine both the number of tables and the number of people from the position number.

(1)-i Investigate the pattern of increase using a table.

s	1	2	3	4
t	4	6	8	10
p	12	16	20	24

The number of table is increasing by 2. The number of people is increasing by 4. Since the number of table is 4 and the number of people is 12 for the first arrangement,

$$t = 4 + 2(s - 1)$$
$$p = 12 + 4(s - 1)$$

If $p = 75$,

$$75 = 12 + 4(s - 1)$$
$$75 - 12 = 4s - 4$$
$$63 + 4 = 4s$$
$$s = 67/4 = 16.75$$

Since s must be a whole number, it must be the 17th arrangement. Therefore, the number of tables is

$$t = 4 + 2(17 - 1)$$
$$t = 4 + 2 \times 16$$
$$t = 4 + 32$$
$$t = 36$$

We need 36 tables.

(1)-i' Calculate with $s = 67/4$ instead of rounding up to a whole number.

$$s = \frac{67}{4}$$
$$t = 4 + 2\left(\frac{67}{4} - 1\right)$$
$$t = 4 + 2 \times \frac{63}{4}$$
$$t = 4 + \frac{63}{2}$$
$$t = \frac{71}{2}$$

Since t must be a whole number, $t = 36$.

○ Students will think about relationships and patterns among quantities and try to make sense of the relationship among the number of tables, the way the tables are arranged and the number of people who can be seated. [Interest, Eagerness, and Attitude]

○ Students will be able to identify quantities that are related to the unknown quantity and solve problems by making use of the relationship. [Mathematical Way of Thinking]

- If any student uses $s = \frac{67}{4}$, have the class think about what s represents. Similarly for t.

(1)-ii Using the table, think about algebraic equations.

s	1	2	3	4
t	4	6	8	10
p	12	16	20	24

Since the number of tables is increasing by 2, $2s$.
The number of people is increasing by 4, $4t$.
When $s = 3$,
the number of table is 8. $8 = 2 \times 3 + \bullet$,
so $\bullet$ must be 2.
The number of people is 20.
$20 = 4 \times 3 + \blacklozenge$, so $\blacklozenge$ must be 8.

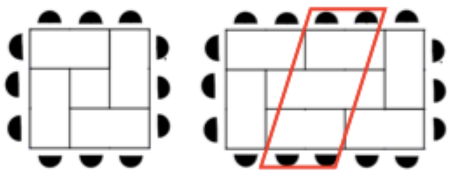
$$t = 2s + 2$$
$$p = 4s + 8$$
$$75 = 4s + 8$$
$$4s = 75 - 8$$
$$4s = 67$$
$$s = \frac{67}{4} = 16.75$$

s must be a whole number. Therefore, it must be the 17th arrangement. So, the number of table is

$$t = 2\times 17 + 2$$
$$t = 34 + 2$$
$$t = 36$$

We need 36 tables.

(1)-iii Focus on the additional tables and people.

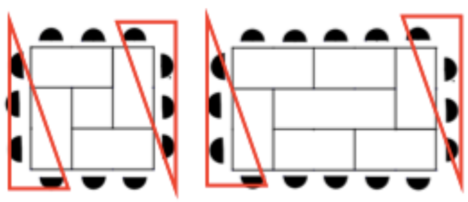


Going from one arrangement to next, we are adding 2 tables and 4 people (shown inside the red box).

$$t = 2(s - 1) + 4$$
$$p = 4(s - 1) + 12$$

(complete the solution similar to (1)-i)

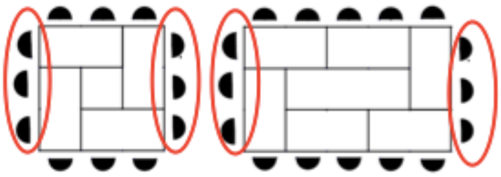
(1)-iv Focus on the unchanged tables and people.



There are 2 pairs of 4 people that remain unchanged (inside the red triangles). Each time, a

pair of 2 people (top and bottom) are added, 2 × 2(s −1)
(complete the solution similar to (1)-i)

(1)-v Think about horizontal sides and vertical sides separately.



Vertical side: there are 1 table on each side that has the longer side vertically (circled in red).
Horizontal side: the number of tables with the longer side horizontal increases by 1 on each side, that is, there are s tables on each side.

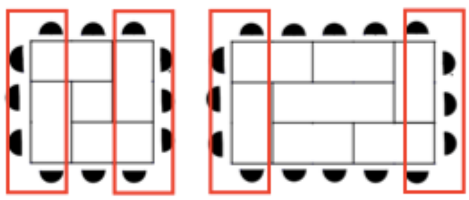
$$t = 1 \times 2 + s \times 2$$

Vertical side: there are 3 people seated on each side (circled in red).
Horizontal side: the number of people increases by 2, that is there are (1 + 2s) people on each side.

$$p = 3 \times 2 + (1 + 2s) \times 2$$

(complete the solution similar to (1)-i)

(1)-v' Think about horizontal sides and vertical sides separately in a different way.



Vertical tables: there are 2 vertical tables on each side always (inside the red boxes).
Horizontal tables: increasing by 1, that is, there are (s − 1) table on each side.

$$t = 2 \times 2 + (s - 1) \times 2$$

Number of people on vertical sides: there are 5 people on each vertical side always (inside the red boxes).
Number of people on horizontal sides: increasing by 2, that is, there are (1 + 2(s − 1) people on each side.

$$p = 5 \times 2 + (1 + 2(s - 1)) \times 2$$

(complete the solution similar to (1)-i)

(2) Consider the position number as an independent variable and the number of tables and the number of people as dependent variables, then derive algebraic equations to determine the number of people and the number of table from the position number.

$t = 2s + 2$

$p = 4s + 8$

$2s + 2 = t$

Therefor, $p = 2t + 4$

Substitute $t = 2s + 2$ in this last equation,

$p = 2(2s + 2) + 4$

(complete the solution similar to (1)-i)

(3) Derive the equation relating the number of table and the number of people directly.

(3)-i Using a table, derive an equation.

s	1	2	3	4
t	4	6	8	10
p	12	16	20	24

If we let x be the increase in the number of tables,

$12 + 2x = 75$

$x = 63/2 = 31.5$

$32 + 4 = 36$

We need 36 tables.

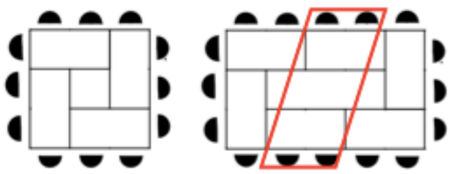
(3)-i' Calculate without using an equation.

(72 − 12) ÷ 2 = 31 rem. 1

The increase in the number of tables is 32.

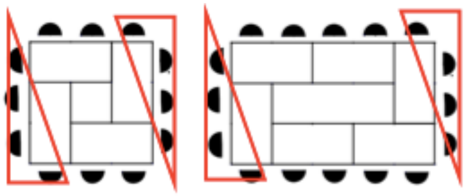
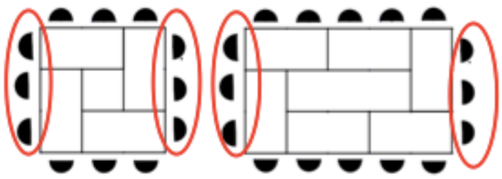
There were 4 tables at first, we need 36 tables.

(3)-iii Focus on the additional tables and people.



As the position number increases by 1, the number of tables will increase by 2 and the number of people by 4 (inside the red box).

The same equation as (3)-i.

	(3)-iv Focus on the unchanged tables and people.  There are 2 pairs of 4 people that remain unchanged (inside the red triangles). Each time, a pair of 2 people (top and bottom) are added. If we say that there are x horizontal tables on each side, so there are $2 \times x$ horizontal tables. $75 = 4 \times 2 + 2 \times x$ $2x = 75 - 8$ $x = 33.5$ Since the number of tables must be a whole number, there are 34 horizontal tables. By adding 1 table each vertical table, there will be 36 tables. (4)-v Think about horizontal sides and vertical sides separately. 	
[Sharing solutions]	Vertical side: there are 3 people seated on each side (circled in red). Horizontal side: the number of people on each side is an odd number and increases by 2 each time a table is added, that is, there are $2x + 1$ people on each horizontal side. $75 = 3 \times 2 + (2x + 1) \times 2$ $4x = 67$ $x = 67/4 = 16.25$	# Make sure students pay careful attention here since the way the letter is used in this approach is different from others up to now.
	There are 17 horizontal tables on each side. $1 \times 2 + 17 \times 2 = 36$ We need 36 tables. (4) Use diagrams. (5) Think about the tables arranged in different ways. • Arrange the tables closer to a square. • Arrange the tables in a step-like arrangement. • By checking different arrangements, try to explain the number of tables needed does not depend on the arrangement.	# If any students make comments like, "my idea is similar to ___'s" or "my idea is different from ____'s," ask students to elaborate where their ideas are similar or different. Have the class think about how algebraic expressions and what is shown on diagrams are related.

		# As students share their ideas, have them articulate details such as whether or not they used equations even though the same diagram was used or the difference in the way letters (variables) are used. # Make sure students pay attention to the distinction of the multiplicand and the multiplier.
[Comparison/ Analysis] [Summary] "What and where was the difference between your idea and other students' ideas?"	• I didn't even imagine we needed 36 tables no matter how they were arranged. • I liked the use of the position number because that made the meaning of each calculation clearer.	# Summarize based on students' solutions and the whole class discussion flexibly.